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ON THE RANGE OF THE DERIVATIVE OF GÂTEAUX-SMOOTH FUNCTIONS ON SEPARABLE BANACH SPACES

BY

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ABSTRACT

We prove that there exists a Lipschitz function from ℓ^1 into $\mathbb{R}^2$ which is Gâteaux-differentiable at every point and such that for every $x, y \in \ell^1$, the norm of $f'(x) - f'(y)$ is bigger than 1. On the other hand, for every Lipschitz and Gâteaux-differentiable function from an arbitrary Banach space X into $\mathbb{R}$ and for every $\varepsilon > 0$, there always exist two points $x, y \in X$ such that $\|f'(x) - f'(y)\|$ is less than ε. We also construct, in every infinite dimensional separable Banach space, a real valued function f on X, which is Gâteaux-differentiable at every point, has bounded non-empty support, and with the properties that f' is norm to weak* continuous and $f'(X)$ has an isolated point a, and that necessarily $a \neq 0$.

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1. Introduction

Let f be a mapping from a Banach space X into a Banach space Y which is Gâteaux-differentiable at every point. Our purpose is the study of the range of the derivative of f. We denote this range $f'(X)$.Let us recall that sufficient conditions on a subset A of a dual Banach space X^* so that it is the range of the derivative of a real valued function on X which is Fréchet-differentiable at each point have been obtained in [BFKL], [BFL], [AFJ] and [G1]. In this case, it was noticed in [AD] that whenever X is an infinite dimensional Banach space with separable dual, there exists a $\mathcal{C}^1$-smooth real valued function on X with bounded support and such that $f'(X) = X^*$. On the other hand, it follows from [H] that if f is a function on c_0 with locally uniformly continuous derivative, then $f'(c_0)$ is included in a countable union of norm compact subsets of ℓ^1. The structure of the range of f' whenever f' satisfies a Hölder condition has been investigated in [G2]. On the other hand, it was observed in [ADJ] that if X and Y are separable Banach spaces and if X is infinite dimensional, one can always find a Gâteaux-differentiable function $f\colon X \to Y$ such that $f'(X)$ coincides with $\mathcal{L}(X, Y)$. We shall investigate here phenomena which can occur when f is Gâteaux-differentiable, but not when f is Fréchet-differentiable. In particular, for each infinite dimensional separable Banach space X, we shall construct in section 2 a Gâteaux-differentiable function f on X, with bounded support, and such that for all $x \neq 0$, $\|f'(x) - f'(0)\| \geq 1$. In section 3, we shall consider the following question: let X, Y be two Banach spaces. Is it possible to construct a Lipschitz continuous mapping $f\colon X \to Y$, Gâteaux-differentiable at each point, and such that, for all $x, y \in X$, $x \neq y$, we have $\|f'(x) - f'(y)\| \geq 1$? Clearly, this is not possible whenever $\mathcal{L}(X, Y)$ is separable. We shall prove that this is not possible either whenever $Y = \mathbb{R}$, but such a construction will be carried out whenever $(X, Y) = (\ell^1, \mathbb{R}^2)$ and whenever $(X, Y) = (\ell^p, \ell^q)$ with $1 \leq p \leq q < +\infty$.

2. Isolated points in the range of the derivative of a function

Let X be a Banach space, and f be a real valued function defined on X. If f is Fréchet-differentiable at every point, then Malý's Theorem ([M]) asserts that the range of f', denoted $f'(X)$, is connected. Therefore, if f is not affine, $f'(X)$, endowed with the norm-topology, has no isolated points. If f is Gâteaux-differentiable at every point of X and if f is not affine, the following proposition says that $f'(X)$ has no w^*-isolated points. We shall see later that in this case $f'(X)$ can have isolated points for the norm topology.

PROPOSITION: *Let X be an infinite dimensional Banach space, and let f be a real valued locally Lipschitz and Gâteaux-differentiable function on X. Then either f is affine or $f'(X)$ has no w^*-isolated points.*

Remark: J. Saint Raymond ([S]) constructed a mapping f from $\mathbb{R}^2$ into $\mathbb{R}^2$, Fréchet-differentiable at each point, and so that $\{det(f'(x)); x \in \mathbb{R}^2\} = \{0,1\}$. Therefore $f'(\mathbb{R}^2)$ is not connected and has two isolated points. Consequently, there is no analog of Malý's theorem and of the above proposition for vector valued mappings.

Proof: Let f be a real valued locally Lipschitz and Gâteaux-differentiable function on X which is not affine. Therefore, $Card(f'(X)) \geq 2$. Fix $y^* \in f'(X)$, and we may assume that $y^* = f'(0)$. To see that y^* is not w^* isolated, fix any $z^* = f'(z) \neq y^*$ and a neighbourhood

$$V = \{x^* \in X^*; |(x^* - y^*)(x_i)| < \varepsilon, 1 \leq i \leq n\}.$$

Without loss of generality, we can assume that $z^* \notin V$ and $x_1 = z$. Define $F\colon \mathbb{R}^n \to \mathbb{R}$ by

$$F(t_1, t_2, \ldots, t_n) = f\left(\sum_{i=1}^{n} t_i x_i\right).$$

Since F is locally Lipschitz continuous and Gâteaux-differentiable on $\mathbb{R}^n$, it is Fréchet-differentiable on $\mathbb{R}^n$ and

$$F'(t_1, t_2, \ldots, t_n) = \left(f'\left(\sum_{i=1}^{n} t_i x_i\right)(x_j)\right)_{j=1}^{n}.$$

So $F'(0,0,\ldots,0) = (\langle y^*, x_i\rangle)$ and $F'(1,0,\ldots,0) = (\langle z^*, x_i\rangle)$. Since $z^* \notin V$, $F'(1,0,\ldots,0) \neq F'(0,0,\ldots,0)$. By Malý's theorem, $F'(\mathbb{R}^n)$ is connected. Therefore, there is a $t = (t_1, t_2, \ldots, t_n)$ such that $0 < \|F'(t) - F'(0)\| < \varepsilon$. Thus, if we set $x = t_1x_1 + t_2x_2 + \cdots + t_nx_n$, we see that $x^* = f'(x) \in V$ and $x^* \neq y^*$.

From now on, we say that a real valued function on an infinite dimensional Banach space X is a **bump** function if it has bounded non-empty support. We shall denote $B(r)$ the set of all $x^* \in X^*$ such that $\|x^*\| < r$. If E is a Banach space, $x \in E$ and $r > 0$, we denote $B_E(x,r)$ (resp. $\overline{B}_E(x,r)$) the open ball (resp. closed ball) in E of center x and radius r. If f is a continuous and Gâteaux-differentiable bump function on X, then, according to the Ekeland variational principle, the **norm** closure of $f'(X)$ contains a ball $B(r)$ for some $r > 0$. A natural conjecture would be that the norm closure of $f'(X)$ is norm connected, or at least that $f'(X)$ does not contain an isolated point. This is not so as shown by the following construction.

THEOREM 1: *Let X be an infinite dimensional separable Banach space. Then, there exists a bump function f on X such that f is Gâteaux-differentiable at every point, f' is norm to weak* continuous and $\|f'(0) - f'(x)\| \geq 1$ whenever $x \neq 0$. If X^* is separable, we can assume moreover that f is C^1 on $X\backslash\{0\}$.*

Remark: According to the above discussion, 0 is not an isolated point of $f'(X)$, so necessarily $f'(0) \neq 0$.

Proof: We shall use two lemmas.

LEMMA 1: *Let X be a Banach space, U be an open connected subset of X^* such that $0 \in U$ and $x^* \in U$. Assume there exists on X a Lipschitz continuous bump function which is Gâteaux-differentiable (resp. Fréchet-differentiable) at every point. Then there exists a Lipschitz continuous bump function β on X with support contained in the unit ball, which is Gâteaux-differentiable (resp. Fréchet-differentiable) at every point, such that $\beta'(X) \subset U$ and $\beta'(x) = x^*$ for all x in a neighbourhood of 0.*

Proof of Lemma 1: Let b be a Lipschitz bump function on X which is Gâteaux-differentiable (resp. Fréchet-differentiable) at every point of X. By translation, we can assume that $b(0) \neq 0$. Replacing $b(x)$ by $\lambda_1 b(\lambda_2 x)$, we can also assume that there exists $0 < \delta < 1$ such that $b(x) \geq 1$ whenever $\|x\| \leq \delta$ and that the support of b is included in the unit ball. Composing b with a suitable C^∞-smooth function from $\mathbb{R}$ into $\mathbb{R}$,we can assume moreover that $b(x) = 1$ whenever $\|x\| \leq \delta$, and that $0 \leq b(x) \leq 1$ for all $x \in X$. Since U is connected, there exists finitely many points $x_0^*, x_1^*, \ldots, x_n^* \in U$ such that $x_0^* = 0$, $x_n^* = x^*$, and the segments $[x_i^*, x_{i+1}^*]$ are included in U. The polygonal line $R = \bigcup_{i=0}^{n-1}[x_i^*, x_{i+1}^*]$ is compact, therefore there exists $\varepsilon > 0$ such that $R + B(\varepsilon) \subset U$. By adding if necessary points on the polygonal line R, we can assume that for all $i \in \{1, 2, \ldots, n\}$, $\|x_i^* - x_{i-1}^*\| < \varepsilon/\|b'\|_\infty$. Define

$$b_i(x) = b(x).(x_i^* - x_{i-1}^*)(x).$$

We have $b_i'(x) = (x_i^* - x_{i-1}^*)(x).b'(x) + b(x).(x_i^* - x_{i-1}^*)$, with $b(x).(x_i^* - x_{i-1}^*) \in [0, x_i^* - x_{i-1}^*]$ and $\|(x_i^* - x_{i-1}^*)(x).b'(x)\| < \varepsilon$ for all $x \in X$, therefore $b_i'(X) \subset [0, x_i^* - x_{i-1}^*] + B(\varepsilon)$. Finally, set

$$\beta(x) = \sum_{i=1}^{n} \delta^{i-1} b_i(x/\delta^{i-1});$$

β is a Lipschitz continuous bump function on X which is Gâteaux-differentiable (resp. Fréchet-differentiable) at every point. Let $x \in X$ and assume that $\delta^i < \|x\| \leq \delta^{i-1}$ for $1 \leq i \leq n$. If $j > i$, $\|x/\delta^{j-1}\| > 1$, so $b_j(y/\delta^{j-1}) = 0$ for all y in a neighbourhood of x and $b_j'(x/\delta^{j-1}) = 0$. If $j < i$, $\|x/\delta^{j-1}\| \leq \delta$, so $b_j'(x/\delta^{j-1}) = x_j^* - x_{j-1}^*$. Therefore

$$\beta'(x) = \sum_{j=1}^{i-1}(x_j^* - x_{j-1}^*) + b_i'(x/\delta^i) = x_{i-1}^* + b_i'(x/\delta^i) \in [x_{i-1}^*, x_i^*] + B(\varepsilon).$$

Moreover, if $\|x\| \leq \delta^n$, then $\beta'(x) = x_n^* = x^*$. Thus $\beta'(x) = x^*$ for all x in a neighbourhood of 0 and $\beta'(X) \subset R + B(\varepsilon) \subset U$.

LEMMA 2: *Let X, Y be two Banach spaces, $a \in X$, V be an open neighbourhood of a, and $f : V \to Y$ be continuous on V and Gâteaux-differentiable at every point of $V\backslash\{a\}$. If $f'(x)$ has a limit ℓ in $\mathcal{L}(X,Y)$ endowed with the strong operator topology as x tends to a, then f is Gâteaux-differentiable at a and $f'(a) = \ell$.*

Proof of Lemma 2: This result is well-known whenever X is the real line. In the general case, fix $h \in X$. The mapping ϕ_h defined on the real line by $\phi_h(t) = f(a + th)$ whenever $t \neq 0$, $\phi_h'(t) = f'(a + th).h$ tends to $\ell.h$ in Y as t tends to 0. Using the one dimensional case, f is differentiable at a in the direction h and $f'(a).h = \ell.h$. This proves that$f$ is Gâteaux-differentiable at a and $f'(a) = \ell$.

In order to prove the theorem, let $a^* \in X^*$ such that $1 < \|a^*\| < 2$. Let (u_n) be a dense sequence in X and

$$V_n = \{x^* \in X^*; |x^*(u_i) - a^*(u_i)| < 1/2^n \text{ for all } i \in \{1, \ldots, n\}\}.$$

Let $(V_n)_{n\geq 0}$ be a decreasing sequence of weak* open subsets containing a so that, if $y_n^* \in V_n$ and if (y_n^*) is bounded, then (y_n^*) converges to a^* for the weak*-topology. Moreover, $W_n = V_n \cap \{x^* \in X^*; 1 < \|x^* - a^*\| < 2\}$ is connected for each n, because X is infinite dimensional. Let $(x_n^*) \subset X^*$ be a sequence such that $x_1^* = 0$ and for every n, $x_n^* \in W_n$. For each n, $1 < \|x_n^* - a^*\| < 2$ and (x_n^*) converges to $a^*$for the weak* topology. $W_n - x_n^* = \{x^* - x_n^*; x^* \in W_n\}$ is a norm open connected subset of X^* containing 0. Since $x_{n+1}^* \in W_{n+1} \subset W_n$, we also have $x_{n+1}^* - x_n^* \in W_n - x_n^*$. Since X is separable (resp. X^* is separable) there exists on X a Lipschitz continuous bump function which is Gâteaux-differentiable (resp. Fréchet-differentiable) at each point. According to Lemma 1, there exists a Lipschitz continuous bump b_n which is Gâteaux-differentiable (resp. Fréchet-differentiable) at every point, such that $b'_n(X) \subset W_n - x^*_n$, with support in the unit ball and such that $b'_n(x) = x^*_{n+1} - x^*_n$ for all x satisfying $\|x\| < \delta_n$. Denote $c_1 = 1$ and, for $n \geq 2$, $c_n = \prod_{i=1}^{n-1} \delta_n$. Define

$$b(x) = \sum_{n=1}^{+\infty} c_n b_n(x/c_n);$$

b has bounded support since $b(x) = 0$ whenever $\|x\| \geq 1$. On $X \backslash \{0\}$ this sum is locally finite, so b is Gâteaux-differentiable (resp. Fréchet-differentiable) at each point of $X\backslash\{0\}$. If $c_{n+1} \leq \|x\| < c_n$, then we have $b'(x) = x^*_n + b'_n(x/c_n) \in W_n$, so $\|b'(x)\|$ is uniformly bounded in x, $b'(X\backslash\{0\}) \subset X^*\backslash B(a^*, 1)$, and $b'(x) \xrightarrow{w^*} a^*$ as $x \to 0$. Lemma 2 then shows that b is Gâteaux-differentiable at 0 and that $b'(0) = a^*$.

3. Can all the derivatives be far away from each other?

We first notice that, under mild regularity assumptions, the answer to the above question is negative for real valued functions.

PROPOSITION: *Let X be a Banach space and $f\colon X \to \mathbb{R}$ be a Lipschitz continuous, everywhere Gâteaux-differentiable function. Then, for every $x \in X$ and every $\varepsilon > 0$, there exists $y, z \in B_X(x, \varepsilon)$ such that $\|f'(y) - f'(z)\| \leq \varepsilon$.*

Proof: We shall actually show that if $f\colon X \to \mathbb{R}$ is locally uniformly continuous and everywhere Gâteaux-differentiable, then, for every $x \in X$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $h \in X$, $\|h\| \leq \delta$, there exists $y \in B_X(x, \varepsilon)$ such that $\|f'(y + h) - f'(y)\| \leq \varepsilon$. Fix $x \in X$ and $\varepsilon_0 > 0$ such that f is uniformly continuous on $B_X(x, 2\varepsilon_0)$. Fix also $0 < \varepsilon < \varepsilon_0$. By uniform continuity, there exists $\delta > 0$ such that $|f(z) - f(y)| < \varepsilon^2/4$ whenever $y, z \in B_X(x, 2\varepsilon_0)$ and $\|z - y\| \leq \delta$. Without loss of generality, we can assume that $\delta < \varepsilon/2$. Take any $h \in X$ such that $\|h\| \leq \delta$. Define $\varphi\colon X \to \mathbb{R}$ by $\varphi(y) = f(y+h) - f(y)$ if $\|y - x\| \leq \varepsilon_0$ and $\varphi(y) = +\infty$ otherwise. The function φ is lower semi-continuous on X and, for all $y \in B_X(x, \varepsilon_0)$, $-\varepsilon^2/4 < \varphi(y) < \varepsilon^2/4$. In particular, $\varphi(x) < \inf_{y \in X} \varphi(y) + \varepsilon^2/2$. The Ekeland variational principle then tells us the existence of $y \in X$ such that $\|y - x\| \leq \varepsilon/2$ and for all $u \in X$, $\varphi(u) \geq \varphi(y) - \varepsilon\|u - y\|$. Since $\|y - x\| \leq \varepsilon/2 < \varepsilon_0$, the function φ is Gâteaux-differentiable at y and we obtain $\|\varphi'(y)\| \leq \varepsilon$. Hence, if we denote $z = y + h$, $\|f'(y) - f'(z)\| \leq \varepsilon$, and we have $\|z - x\| \leq \|h\| + \|y - x\| < \varepsilon$.

The derivatives of a Fréchet-differentiable mapping cannot be far away from each other for mappings which are everywhere Fréchet-differentiable.

PROPOSITION: *Let X, Y be separable Banach spaces and $f\colon X \to Y$ be an everywhere Fréchet-differentiable locally uniformly continuous mapping. Then, for every $x \in X$ and every $\varepsilon > 0$, there exists $y, z \in B_X(x,\varepsilon)$, $y \neq z$, such that $\|f'(y) - f'(z)\| \leq \varepsilon$.*

Proof: Fix $\varepsilon > 0$ and $n_0 > 0$ such that f is uniformly continuous on $B_X(x, \varepsilon + 1/n_0)$. For each $n \geq 1$, define

$$A_n = \{y \in B_X(x,\varepsilon), \|f(y+h) - f(y) - f'(y).h\| \leq \varepsilon\|h\| \text{ whenever } \|h\| \leq 1/n\}.$$

Since $B_X(x,\varepsilon) = \bigcup_{n \geq n_0} A_n$, there exists $n_1 \geq n_0$ and $u \in B_X(x,\varepsilon)$ such that u is an accumulation point of A_{n_1}. Pick $y, z \in A_{n_1}$ such that $y \neq z$ and $\|y - z\| < \alpha$, where α is chosen so that $\|f(u) - f(v)\| \leq \varepsilon/n_1$ whenever $u, v \in B(x, \varepsilon + 1/n_0)$ and $\|u - v\| < \alpha$. We have

$$\|f(y+h) - f(y) - f'(y).h\| \leq \varepsilon/n_1 \quad \text{and} \quad \|f(z+h) - f(z) - f'(z).h\| \leq \varepsilon/n_1$$

for all h such that $\|h\| \leq 1/n_1$, so

$$\|(f(y+h) - f(z+h)) - (f(y) - f(z)) - (f'(y) - f'(z)).h\| \leq 2\varepsilon/n_1.$$

Therefore,

$$\|(f'(y) - f'(z)).h\| \leq 4\varepsilon/n_1.$$

Since this is satisfied for all h such that $\|h\| \leq 1/n_1$, we obtain that $\|f'(y) - f'(z)\| \leq 4\varepsilon$.

In view of the above propositions, one could believe that whenever X, Y are Banach spaces (or vector normed spaces) and $f\colon X \to Y$ is a mapping Gâteaux-differentiable at each point of X, then for every $\varepsilon > 0$, there exists $y, z \in X$ such that $\|f'(y) - f'(z)\| \leq \varepsilon$. Our next result proves that this is not so.

THEOREM 2: (1) *There exists a Lipschitz mapping $F\colon \ell^1 \to \mathbb{R}^2$, Gâteaux-differentiable at each point of ℓ^1, such that for every $x, y \in \ell^1$, $x \neq y$, then $\|F'(x) - F'(y)\|_{\mathcal{L}(\ell^1,\mathbb{R}^2)} \geq 1$. Moreover, for each $h \in \ell^1$, $x \to F'(x).h$ is continuous from ℓ^1 into $\mathbb{R}^2$.*

(2) *Let us denote D the vector normed space of elements of ℓ^1 with finite support. There exists a Lipschitz function $G\colon \ell^1 \to \mathbb{R}$, Gâteaux-differentiable at each point of ℓ^1, such that for every $x, y \in D$, $x \neq y$, then $\|G'(x) - G'(y)\|_{\ell^\infty} \geq 1$.*

We shall construct F and G with the properties of Theorem 2 using series. We were inspired by a construction from [DI]. We need an auxiliary construction.

LEMMA 3: *Given $\Delta = (a', a, b, b') \in \mathbb{R}^4$ such that $a' < a < b < b'$ and $\varepsilon > 0$, there exists a $\mathcal{C}^\infty$-function $\varphi = \varphi_{\Delta,\varepsilon}\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that:*

(i) $|\varphi(x,y)| \leq \varepsilon$ *for all* $(x,y) \in \mathbb{R}^2$,
(ii) $\varphi(x,y) = 0$ *whenever* $x \notin [a', b']$,
(iii) $\|\frac{\partial \varphi}{\partial x}(x,y)\| \leq \varepsilon$ *for all* $(x,y) \in \mathbb{R}^2$,
(iv) $\|\frac{\partial \varphi}{\partial y}(x,y)\| = 1$ *whenever* $x \in [a,b]$,
(v) $\|\frac{\partial \varphi}{\partial y}(x,y)\| \leq 1$ *for all* $(x,y) \in \mathbb{R}^2$,
(vi) *if we denote* $\varphi(x,y) = (\varphi_1(x,y), \varphi_2(x,y))$, *then* $\frac{\partial \varphi_1}{\partial y}(x,0) = 1$ *whenever* $x \in [a,b]$.

Proof of Lemma 3: Let $\beta\colon \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$-smooth function such that $0 \leq \beta(x) \leq 1$ for all x, $\beta(x) = 0$ whenever $x \notin [a', b']$ and $\beta(x) = 1$ whenever $x \in [a,b]$. If $n \geq 1$ is large enough, the function defined by

$$\varphi(x,y) = \frac{\beta(x)}{n}(\sin(ny), \cos(ny))$$

satisfies the desired properties.

We shall also use the following criterion of Gâteaux-differentiability of the sum of a series:

LEMMA 4: *Let X and Y be Banach spaces and, for all n, let $f_n\colon X \to Y$ be Gâteaux-differentiable mappings. Assume that $(\sum f_n)$ converges pointwise on X, and that there exists a constant $K > 0$ so that for all h,*

$$\sum_{n \geq 1} \sup_{x \in X} \left\| \frac{\partial f_n}{\partial h}(x) \right\| \leq K\|h\|. \tag{1}$$

Then the mapping $f = \sum_{n\geq 1} f_n$ is Gâteaux-differentiable on X, for all x, $f'(x) = \sum_{n \geq 1} f_n'(x)$ (where the convergence of the series is in $\mathcal{L}(X,Y)$ for the strong operator topology), and f is K-Lipschitz. Moreover, if each f_n' is continuous from X endowed with the norm topology into $\mathcal{L}(X,Y)$ with the strong operator topology, then f' shares the same continuity property.

Proof of Lemma 4: Fix $x \in X$. First observe that condition (1) implies that for all h, the series $(\sum \frac{\partial f_n}{\partial h}(x)) = (\sum f_n'(x).h)$ converges in Y. Therefore, the series $(\sum f_n'(x))$ converges in $\mathcal{L}(X,Y)$ for the strong operator topology, to some operator $T \in \mathcal{L}(X,Y)$, and by (1), $\|T\| \leq K$. For each $h \in X$, we define $g_n\colon \mathbb{R} \to Y$ by $g_n(t) = f_n(x + th)$. The function $g = \sum_{n \geq 1} g_n$ is well defined. Since

$$\sum_{n\geq 1} \|g_n'\|_\infty \leq \sum_{n \geq 1} \sup_{x \in X} \left\| \frac{\partial f_n}{\partial h}(x) \right\| \leq K\|h\|$$

the mapping g is differentiable and

$$g'(0) = \sum_{n\geq 1} g'_n(0) = \sum_{n\geq 1} \frac{\partial f_n}{\partial h}(x) = T(h).$$

Thus we have proved that f is differentiable along every direction h and that $\frac{\partial f}{\partial h}(x) = T(h)$. In other words, f is Gâteaux-differentiable at x and $f'(x) = T$. Since for all x, $\|f'(x)\| \leq K$, the mean value theorem implies that f is K-Lipschitz.

Proof of Theorem 2, part (1): Fix an enumeration $\Delta_k = (a'_k, a_k, b_k, b'_k)$, $k \in N$, of all quadruples of dyadic numbers such that $a'_k < a_k < b_k < b'_k$. Select integers m_k^n such that for each n, $n < m_k^n$ and $(m_k^n)_k$ is an increasing sequence, and satisfying

$$m_k^n = m_\ell^p \Rightarrow n = p \text{ and } k = \ell. \tag{2}$$

This condition is satisfied, for instance, whenever $m_k^n = 2^k.3^n$. Fix $\varepsilon > 0$ and let ε_k^n be positive real numbers such that $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \varepsilon_k^n = \varepsilon$. We note $\varepsilon_k = \sum_{n=1}^{\infty} \varepsilon_k^n$, so that $\sum_{k=1}^{\infty} \varepsilon_k = \varepsilon$. Put $f_{n,k}\colon \ell^1 \to \mathbb{R}^2$ such that, if $x = (x_i) \in \ell^1$, then $f_{n,k}(x) = \varphi_{\Delta_k,\varepsilon_k^n}(x_n, x_{m_k^n})$:$f_{n,k}$ is a $\mathcal{C}^\infty$ function on ℓ^1. The function $F\colon \ell^1 \to \mathbb{R}^2$ we are looking for is defined by

$$F(x) = \sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}} f_{n,k}(x).$$

CLAIM 1: *F is well-defined.*

Indeed, according to condition (i) of Lemma 3, $\|f_{n,k}\|_\infty = \|\varphi_{\Delta_k,\varepsilon_k^n}\|_\infty = \varepsilon_k^n$, so the series defining F converges uniformly.

CLAIM 2: *F is Gâteaux-differentiable on ℓ^1 and F is $(1+\varepsilon)$-Lipschitz-continuous on ℓ^1.*

To see this, we apply Lemma 4: let $h = (h_1, \ldots, h_n, \ldots) \in \ell^1$. By (iii) and (v), we have for all n, k

$$\sup_{x\in X}\left\|\frac{\partial f_{n,k}}{\partial h}(x)\right\| \leq |h_{m_k^n}| + \varepsilon_k^n|h_n| \leq |h_{m_k^n}| + \varepsilon_k^n\|h\|_1.$$

So, because of condition (2),

$$\sum_{n,k}\sup_{x\in X}\left\|\frac{\partial f_{n,k}}{\partial h}(x)\right\| \leq (1+\varepsilon)\|h\|_1.$$

We have proved that condition (1) of Lemma 4 is satisfied with $K = 1+\varepsilon$, thus F is Gâteaux-differentiable on ℓ^1 and F is $(1+\varepsilon)$-Lipschitz-continuous on ℓ^1.

CLAIM 3: *If $x \neq y \in \ell^1$, then $\|F'(x) - F'(y)\|_{\mathcal{L}(\ell^1,\mathbb{R}^2)} \geq 1 - 2\varepsilon$.*

Indeed, let $n \in \mathbb{N}$ such that $x_n \neq y_n$. Let k be such that $x_n \in [a_k, b_k]$ and $y_n \notin [a'_k, b'_k]$. According to (ii) and (iv) of Lemma 3,

$$\left\|\frac{\partial f_{n,k}}{\partial x_{m_k^n}}(x)\right\| = 1 \quad \text{and} \quad \frac{\partial f_{n,k}}{\partial x_{m_k^n}}(y) = 0.$$

On the other hand, for all r,

$$\left\|\frac{\partial f_{m_k^n,r}}{\partial x_{m_k^n}}(x)\right\| \leq \varepsilon_r \quad \text{and} \quad \left\|\frac{\partial f_{m_k^n,r}}{\partial x_{m_k^n}}(y)\right\| \leq \varepsilon_r$$

and, if $\ell \neq m_k^n$ and $(\ell, r) \neq (n, k)$,

$$\frac{\partial f_{\ell,r}}{\partial x_{m_k^n}}(x) = 0 \quad \text{and} \quad \frac{\partial f_{\ell,r}}{\partial x_{m_k^n}}(y) = 0.$$

Therefore,

$$\begin{aligned}\|F'(x) - F'(y)\|_{\mathcal{L}(\ell^1,\mathbb{R}^2)} &\geq \left\|\frac{\partial F}{\partial x_{m_k^n}}(x) - \frac{\partial F}{\partial x_{m_k^n}}(y)\right\| \\ &\geq 1 - \sum_{(\ell,r)\neq(n,k)} \left\|\frac{\partial f_{\ell,r}}{\partial x_{m_k^n}}(x) - \frac{\partial f_{\ell,r}}{\partial x_{m_k^n}}(y)\right\| \\ &\geq 1 - 2\varepsilon.\end{aligned}$$

Let us now prove part (2) of Theorem 2. Since $F\colon \ell^1 \to \mathbb{R}^2$, we can write $F = (G, H)$, where $G, H\colon \ell^1 \to \mathbb{R}$. We shall also denote $f_{n,k} = (g_{n,k}, h_{n,k})$; $G\colon \ell^1 \to \mathbb{R}$ is Lipschitz continuous, Gâteaux-differentiable at each point of ℓ^1. Let $x = (x_i)$, $y = (y_i) \in D$ and n be such that $x_n \neq y_n$. Let k be such that $x_n \in [a_k, b_k]$, $y_n \notin [a'_k, b'_k]$ and $x_{m_k^n} = 0$. According to (vi) of Lemma 3, we have

$$\left\|\frac{\partial g_{n,k}}{\partial x_{m_k^n}}(x)\right\| = 1 \quad \text{and} \quad \frac{\partial g_{n,k}}{\partial x_{m_k^n}}(y) = 0.$$

We conclude, as in the proof of Claim 3 of part (1), that

$$\|G'(x) - G'(y)\|_{\ell^\infty} \geq 1 - 2\varepsilon.$$

Remark: (1) If we set $\Phi = f/(1 - 2\varepsilon)$, we have obtained, for every $\alpha > 0$, the construction of a function $\Phi\colon \ell^1 \to \mathbb{R}^2$, Gâteaux-differentiable at every point of ℓ^1, satisfying:

(i) for all $x, y \in \ell^1$, $\|\Phi(x) - \Phi(y)\| \leq (1 + \alpha)\|x - y\|_1$,

(ii) for all $x \neq y \in \ell^1$, $\|\Phi'(x) - \Phi'(y)\|_{\mathcal{L}(\ell^1,\mathbb{R}^2)} \geq 1$.

(2) Fix $h \in \ell^1$. Since $x \to F'(x).h$ is continuous from ℓ^1 into $\mathbb{R}^2$, the set $\{F'(x).h; x \in \ell^1\}$ is connected. This is in contrast with the fact that $\{F'(x); x \in \ell^1\}$ is discrete in $\mathcal{L}(\ell^1, \mathbb{R}^2)$.
(3) A careful look at the above construction shows that f is uniformly Gâteaux-differentiable.
(4) Note that for cardinality reasons, if $dim(X) \geq 1$ and $\mathcal{L}(X,Y)$ is separable, then for all Gâteaux-differentiable mappings from X into Y, and for all $\varepsilon > 0$, there exists $y, z \in X$, $y \neq z$ such that $\|f'(y) - f'(z)\| \leq \varepsilon$. Therefore, it is not possible to replace ℓ^1 by ℓ^p $(p > 1)$ in Theorem 2. However, there exists a Lipschitz function $H\colon \ell^2 \to \ell^2$, Gâteaux-differentiable at each point of ℓ^2, such that for every $x, y \in \ell^2$, if $x \neq y$, then

$$\|H'(x) - H'(y)\|_{\mathcal{L}(\ell^2)} \geq 1.$$

This will follow from the following more general result:

THEOREM 3: *Let $X_p = \ell^p$ if $1 \leq p < +\infty$ and $X_\infty = c_0$. Let us fix $1 \leq p, q \leq +\infty$. The following assertions are equivalent:*

1. *There exists a Lipschitz continuous mapping $H\colon X_p \to X_q$, Gâteaux-differentiable at each point of X_p, such that for every $x, y \in X_p$, $x \neq y$, then $\|H'(x) - H'(y)\|_{\mathcal{L}(X_p, X_q)} \geq 1$.*
2. *$p \leq q$.*
3. *$\mathcal{L}(X_p, X_q)$ is not separable.*

Proof of Theorem 3: According to Remark (4) above, (1) implies (3). If $p > q$, then by Pitt's theorem, all operators from X_p to X_q are compact, hence $\mathcal{L}(X_p, X_q)$ is separable. Therefore (3) implies (2). So it remains to prove that (2) implies (1). Assume that $p \leq q$ and let (e_n) be the usual basis of X_p. Let $T_k \in \mathcal{L}(\mathbb{R}^2, X_q)$ defined by $T_k(x,y) = xe_{2k} + ye_{2k+1}$. Denote a_q the common norm of the operators T_k. Let Δ_k, ε_k^n, m_k^n and $\varphi_{\Delta_k, \varepsilon_k^n}$ be defined as in the proof of Theorem 2. Put $f_{n,k}\colon X_p \to X_q$ such that, if $x = (x_i) \in X_p$, then $f_{n,k}(x) = T_{m_k^n} \circ \varphi_{\Delta_k, \varepsilon_k^n}(x_n, x_{m_k^n})$: the function $f_{n,k}$ is a $\mathcal{C}^\infty$ mapping from X_p into X_q. The function $H\colon X_p \to X_q$ we are looking for is defined by

$$H(x) = \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} f_{n,k}(x).$$

As in the proof of Theorem 2, H is well-defined. Lemma 4 is no longer applicable in order to show that H is Gâteaux-differentiable at each point of X_p. But Lemma 4 remains true if the hypothesis (1) from Lemma 4 is replaced by condition (3) below:

$$\text{(3)} \qquad \text{for all } h, \left(\sum \frac{\partial f_n}{\partial h}(x)\right) \text{ converges uniformly with respect to } x.$$

So, fix $h = (h_1, \ldots, h_n, \ldots) \in X_p$. We have

$$\frac{\partial f_{n,k}}{\partial h}(x) = h_n u_{k,n}(x) + h_{m_k^n} v_{k,n}(x)$$

with $\|u_{k,n}(x)\|_q \leq \varepsilon_k^n a_q$, $v_{k,n}(x) \in span\{e_{2m_k^n}, e_{2m_k^n+1}\}$ and $\|v_{k,n}(x)\|_q \leq a_q$.

We claim that both series $(\sum_{k,n} h_n u_{k,n}(x))$ and $(\sum_{k,n} h_{m_k^n} v_{k,n}(x))$ are uniformly converging with respect to x. Indeed, for the first one, this follows from the fact that for each x, $\|h_n u_{k,m}(x)\|_q \leq \|h\|_p . a_q . \varepsilon_k^n$, and that $\sum_{n=1}^\infty \sum_{k=1}^\infty \varepsilon_k^n < +\infty$. For the second one, $(\sum_k h_{m_k^n} v_{k,m}(x))$ converges uniformly because it satisfies the uniform Cauchy condition. Indeed, fix $\delta > 0$ and a finite set $A \subset \mathbb{N} \times \mathbb{N}$ such that $\sum_{(k,n)\notin A} h_{m_k^n}^p < \delta^p$. For fixed x, the $v_{k,n}(x)$ are elements of X_q with disjoint supports, so, for any finite subset F of $(\mathbb{N} \times \mathbb{N})\backslash A$,

$$\begin{aligned}\Big\| \sum_{(n,k)\in F} h_{m_k^n} v_{k,m}(x) \Big\|_{X_q} &= \Big(\sum_{(n,k)\in F} \|h_{m_k^n} v_{k,m}(x)\|_{X_q}^q \Big)^{1/q} \\ &\leq a_q \Big(\sum_{(n,k)\in F} h_{m_k^n}^q \Big)^{1/q} \leq a_q \Big(\sum_{(n,k)\in F} h_{m_k^n}^p \Big)^{1/p} < a_q \delta.\end{aligned}$$

Notice that we used in the above chain of inequalities the fact that $p \leq q$. The above estimate is uniform in x, therefore the series $(\sum_{k,n} h_{m_k^n} v_{k,n}(x))$ satisfies the uniform Cauchy condition. Applying the variant of Lemma 4 mentioned above, we get that H is Lipschitz continuous and Gâteaux-differentiable at each point of X_p. As in the proof of Theorem 2, one sees that there exists $a > 0$ such that for every $x, y \in X_p$, if $x \neq y$, then $\|H'(x) - H'(y)\|_{\mathcal{L}(\ell^p, \ell^q)} \geq a$.

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